# Probability Theory

Textbook: A First Course in Probability, Sheldon Ross, 2019.

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Chapter 01: Combinatorial Analysis



## Introduction

### Combinatorial Analysis

- Many **basic** probability problems are counting problems.
- Combinatorial analysis is the mathematical theory of counting.

### Example

A communication system is to consist of 4 seemingly identical antennas that are to be lined up in a linear order. The resulting system will be able to receive all incoming signals as long as no two consecutive antennas are defective. If it turns out that exactly 2 of the 4 antennas are defective, what is the probability that the resulting system will be functional?

- Possible configurations: (1: working antenna, 0: defective antenna.)
  0 1 1 0   0 1 0 1   1 0 1 0   0 0 1 1   1 0 0 1   1 1 0 0
- Probability = 3/6 = 1/2.





## Basic Principle of Counting

Suppose that two experiments are to be performed. Then if experiment 1 can result in any one of $m$ possible outcomes and if, for each outcome of experiment 1, there are $n$ possible outcomes of experiment 2, then together there are $mn$ possible outcomes of the two experiments.

### Example

A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

### Solution

- $1^{st}$ experiment: choosing a woman $\Rightarrow$ 10 possible ways.
- $2^{nd}$ experiment: choosing one of *her* children $\Rightarrow$ 3 possible ways.
- From the basic principle that there are $10 \times 3 = 30$ possible choices.





## Generalized Basic Principle of Counting

If $r$ experiments that are to be performed are such that the first one may result in any of $n_1$ possible outcomes; and if, for each of these $n_1$ possible outcomes, there are $n_2$ possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are $n_3$ possible outcomes of the third experiment; and if ..., then there is a total of $n_1 \times n_2 \times \cdots \times n_r$ possible outcomes of the $r$ experiments.

### Example

How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

### Solution

By the generalized version of the basic principle, the answer is $26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000$.





## Generalized Basic Principle of Counting (more examples)

### Example

How many different 7-place license plates are possible if the first 3 places are to be occupied by letters, the final 4 by numbers, and repetition among letters or numbers are prohibited?

### Solution

There would be $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$ possible license plates.





## Ordered Sampling with Replacement

If $r$ experiments that are to be performed are such that the first one may result in any of $n$ possible outcomes; and if, for each of these $n$ possible outcomes, there are $n$ possible outcomes of the second experiment; and if, for each of the possible outcomes of the first two experiments, there are $n$ possible outcomes of the third experiment; and if ..., then the number of possible outcomes of the $r$ experiments is

$$n^r$$

### Example

How many functions defined on $n$ points are possible if each functional value is either 0 or 1?

Let the points be $1, 2, \cdots, n$. We have $f = \Big(f(1), f(2), \cdots, f(n)\Big)$, where $f(i) \in \{0, 1\}$. Therefore the number of possible functions is

$$\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$$

## Ordered Sampling without Replacement

**Example**

How many different ordered arrangements of the letters *a*, *b*, and *c* are possible?

By direct enumeration we see that there are 6, namely, *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*. Each arrangement is known as a permutation. Thus, there are 6 possible permutations of a set of 3 objects.

This result could also have been obtained from the basic principle, since the first object in the permutation can be any of the 3, the second object in the permutation can then be chosen from any of the remaining 2, and the third object in the permutation is then the remaining 1. Thus, there are $3 \times 2 \times 1 = 6$ possible permutations.





## Permutations

Suppose that we have $n$ objects. The basic principle of counting shows that the number of permutations of these $n$ objects is given by

$$n(n-1)(n-2)\ldots 3 \times 2 \times 1 = n!$$

**Example**

A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.

1. How many different rankings are possible?
2. If the men are ranked just among themselves and the women just among themselves, how many different rankings are possible?

1. $10! = 3,628,800$ possible rankings.
2. $6! \times 4! = 720 \times 24 = 17,280$ possible rankings.





## Generalized Permutations

**Permutations of Four Distinct Objects**

| | | | | | |
|---|---|---|---|---|---|
| ABCD | ABDC | ACBD | ACDB | ADBC | ADCB |
| BACD | BADC | BCAD | BCDA | BDAC | BDCA |
| CABD | CADB | CBAD | CBDA | CDAB | CDBA |
| DABC | DACB | DBAC | DBCA | DCAB | DCBA |

**If A = B = X**

| | | | | | |
|---|---|---|---|---|---|
| XXCD | XXDC | XCXD | XCDX | XDXC | XDCX |
| XXCD | XXDC | XCXD | XCDX | XDXC | XDCX |
| CXXD | CXDX | CXXD | CXDX | CDXX | CDXX |
| DXXC | DXCX | DXXC | DXCX | DCXX | DCXX |

**If A = B = C = X**

| | | | | | |
|---|---|---|---|---|---|
| XXXD | XXDX | XXXD | XXDX | XDXX | XDXX |
| XXXD | XXDX | XXXD | XXDX | XDXX | XDXX |
| XXXD | XXDX | XXXD | XXDX | XDXX | XDXX |
| DXXX | DXXX | DXXX | DXXX | DXXX | DXXX |





## Generalized Permutations

Suppose that we have $n$ objects, of which $n_1$ are alike, $n_2$ are alike, ..., $n_r$ are alike. The number of permutations of these $n$ objects is

$$\frac{n!}{n_1! \times n_2! \times \cdots \times n_r!}$$

**Example**

A chess tournament has 10 competitors, of which 4 are Russian, 3 are from the United States, 2 are from Great Britain, and 1 is from Brazil. If the tournament result lists just the nationalities of the players in the order in which they placed, how many outcomes are possible?

$$\frac{10!}{4! \times 3! \times 2! \times 1!} = 12,600$$





## Permuted Selection

The number of different ways that a group of $r$ items could be selected from $n$ items when the order of selection is relevant:

$$P(n,r) = {}^{n}P_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-r+1)$$

**Example**

You are asked to perform three-color painting on a new drawing. In how many ways can the drawing be painted if your palette consists of 12 different colors?

$${}^{12}P_3 = \frac{12!}{(12-3)!} = 12 \times 11 \times 10 = 1320$$



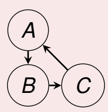


## Circular Permutation

The number of ways to arrange $n$ distinct objects along a fixed circle is

$$\frac{n!}{n} = (n-1)!$$

All cyclic permutations of objects are equivalent because the circle can be rotated.




*ABC* *BCA* *CAB*

**Example**

In how many ways can 3 boys be arranged in a circle?

$2!$

## Unordered Sampling without Replacement

We are interested in determining the number of different groups of $r$ objects that could be formed from a total of $n$ objects.

### Example

How many different groups of 3 could be selected from the 5 items $A$, $B$, $C$, $D$, and $E$?

### Solution

- If the order is relevant, the number of groups is ${}^5P_3$.
- Since every group of 3 (say, $ABC$) will be counted 6 times (that is, all of the permutations $ABC$, $ACB$, $BAC$, $BCA$, $CAB$, and $CBA$ will be counted when the order of selection is relevant), it follows that the total number of groups that can be formed is

$${}^5P_3 \Big/ 3!$$





## Combinations

The number of possible combinations of $n$ objects taken $r$ at a time:

$${}^nC_r = \binom{n}{r} = \frac{n!}{(n-r)!\ r!}$$

### Example

1. From a group of 5 women and 7 men, how many different committees consisting of 2 women and 3 men can be formed?
2. What if 2 of the men are feuding and refuse to serve on the committee together?

1. $\binom{5}{2} \times \binom{7}{3} = 10 \times 35 = 350$ committees.
2. $\binom{5}{2} \times \left(\binom{7}{3} - \binom{2}{2}\binom{5}{1}\right) = 10 \times (35-5) = 300$ committees.





## Combinations (More examples)

### Example

A dance class consists of 10 women and 12 men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

Three experiments: pick women, pick men, and then pair off.

$$\binom{10}{5} \times \binom{12}{5} \times 5! = 23,950,080$$

### Example

In how many ways can 4 boys be chosen from 7 boys be arranged to stand in a circle.

Two experiments: pick boys, and then perform circular permutation.

$$\binom{7}{4} \times (4-1)! = 210$$





## Combinations (More examples)

### Example

Consider a set of $n$ antennas of which $m$ are defective and $n-m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

- Align the $n-m$ functional antennas among themselves.

$$11\cdots1$$

- There are $n-m+1$ spaces between the functional antennas.

$$\wedge 1 \wedge 1 \wedge \cdots \wedge 1 \wedge$$

- Each space may hold at most one of the $m$ defective antenna.
- We just want to pick $m$ spaces among the available $n-m+1$.

$$\binom{n-m+1}{m}$$





## Ordered Partitions

### Multinomial Coefficients

The number of possible divisions of $n$ distinct objects into $r$ distinct groups of respective sizes $n_1, n_2, \cdots, n_r$ is

$$\binom{n}{n_1, n_2, \cdots, n_r} = \frac{n!}{n_1!\ n_2!\ \cdots\ n_r!}$$

### Proof

$$\binom{n}{n_1, n_2, \cdots, n_r} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r}$$

$$= \frac{n!}{(n-n_1)!\ n_1!} \times \frac{(n-n_1)!}{(n-n_1-n_2)!\ n_2!} \times \frac{(n-n_1-n_2)!}{(n-n_1-n_2-n_3)!\ n_3!} \times \cdots \frac{(n-n_1-n_2-\cdots-n_{r-1})!}{0!\ n_r!}$$

$$= \frac{n!}{n_1!\ n_2!\ \cdots\ n_r!}$$





## Ordered and Unordered Partitions

### Example

Ten children are to be divided into an $A$ team and a $B$ team of 5 each. The $A$ team will play in one league and the $B$ team in another. How many different divisions are possible?

$$\binom{10}{5,5} = \frac{10!}{5! \times 5!} = 252 \text{ possible divisions.}$$

### Example

To play a basketball game, 10 children divide themselves into two teams of 5 each. How many different divisions are possible?

The order of the two teams is irrelevant. There is just a division.

$$\binom{10}{5,5} \Big/ 2! = \frac{10!}{5! \times 5!} \Big/ 2! = 126 \text{ possible divisions.}$$

## Unordered Sampling with Replacement

The number of ways to distribute $r$ distinguishable balls among $n$ urns is $n^r$. What if the balls were indistinguishable?

### Equivalent Problem

The number of distinct solutions to the equation

$$x_1 + x_2 + \ldots + x_n = r, \qquad \text{where } x_i \in \mathbb{N}.$$

Map each natural number $x_i$ with $x_i$ vertical lines, i.e.,

$$1 \to | \qquad 2 \to || \qquad 3 \to ||| \qquad \ldots$$

For any solution to the above equation, replacing the $x_i$'s by their mapping would result in a unique representation using $r$ vertical lines ('$|$') and $n-1$ plus signs ('$+$'). For instance, if we have $[x_1, x_2, x_3, x_4]' = [3, 0, 2, 1]'$ as a candidate solution, we shall equivalently get $|||++||+|$.

$$\binom{n+r-1}{r}$$





## Unordered Sampling with Replacement (Examples)

### Example

How many different dominoes are there in a complete set?

Sampling $r = 2$ numbers from $A = \{0, 1, 2, \cdots, 6\}$ with replacement:

$$\binom{n+r-1}{r} = \binom{7+2-1}{2} = \binom{8}{2} = 28.$$

### Example

In how many ways can 10 similar balls be distributed into 3 different urns such that no urn is left empty?

We put a ball in each urn, and then we sample $r = 7$ times among $n = 3$ urns with replacement and place a ball in each sampled urn.

$$\binom{n+r-1}{r} = \binom{3+7-1}{7} = \binom{9}{7} = \binom{9}{2} = 36.$$